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Discretization for Uniform Polynomial Approximation

ROBERT WHITLEY

University of California, Irvine, California 92717, USA

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Let P be the polynomial of degree less than or equal to n which is the best approximation to a given f in $C[-1, 1]$. An approximation to P can be computed by choosing a finite subset F of $[-1, 1]$ and calculating the polynomial P_F, of degree less than or equal to n, which best approximates f on F. Then if $|F|$ (see Eq. (3)) is small, estimates show that the discretization error, as measured by $\|P - P_F\|$, is also small [5, pp. 84–100; 20, pp. 33–47; 22]. A classical choice for the set F of m points is

$$\{\cos((2j-1)\pi/2m): j = 1, 2,..., m\} \tag{1}$$

[5, p. 93].

A natural formulation of this discretization problem, developed below, leads to a specific criterion for the choice of points in F. It will be shown that, by this criterion, the choice of points in (1) is asymptotically best, but not best.

Consider a strictly monotone function ϕ, mapping an interval $[a, b]$ onto $[-1, 1]$, which is continuously differentiable. Then

$$d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)| \tag{2}$$

defines a metric on $[-1, 1]$ which is equivalent to the Euclidean metric. This function $\varnothing$ will play the same role as the function $\cos x$ on $[0, \pi]$ in the classical treatment [5]. If the bound on $|\phi'|$ on $[a, b]$ is M, the mean value theorem shows that $d(x, y) \geqslant (1/M)|x - y|$. If $M > 1$, we can, with no loss of essential generality, consider, instead of ϕ, the function $\phi(x/M)$ on $[aM, bM]$; and so we will suppose that

$$d(x, y) \geqslant |x - y|.$$

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For a subset F of $[-1, 1]$, set

$$|F| = \sup_x \inf_y \{d(x, y): y \text{ in } F, x \text{ in } [-1, 1]\}. \tag{3}$$

For any set G not $[-1, 1]$, let

$$\|g\|_G = \sup_x \{|g(x)|: x \text{ in } G\}$$

and reserve $\|g\|$, without the subscript, for the $C[-1, 1]$ norm. As usual, π_n denotes all the polynomials of degree less than or equal to n, and ω_f is the modulus of continunity of f.

THEOREM 1. *For f in $C[-1, 1]$, let P be the polynomial in π_n which best approximates f (on $[-1, 1]$), and, for a given subset F of $[-1, 1]$, let P_F be the polynomial in π_n which best approximates f on F. Then*

$$\|P - P_F\| \leqslant C \left[\omega_f(\delta) + 2\|f\| \frac{K_n \delta}{1 - K_n \delta} \right] \tag{4}$$

whenever $|F| \leqslant \delta < 1/K_n$. The constant C depends only on f and n, not on ϕ or F. The constant K_n is the norm of the derivative D when restricted to the subspace $\pi_n \circ \phi$ of all functions of the form $Q \circ \phi$, in π_n:

$$K_n = \|D|_{\pi_n \circ \phi}\|. \tag{5}$$

Proof. The proof is similar to [5, pp. 91–92]. By the strong unicity theorem [5, p. 80],

$$\|P - P_F\| \leqslant (1/\gamma)(\|f - P_F\| - \|f - P\|), \tag{6}$$

where γ is a constant which depends on f and n, but not on F (or ϕ). If inequality (4) holds whenever $|F| < \delta$, then it also holds for $|F| = \delta$, so suppose that $|F| < \delta$. There is a point x in $[-1, 1]$ at which $|f(x) - P_F(x)| = \|f - P_F\|$, and a point y in F with $d(x, y) < \delta$. Write

$$\|f - P_F\| \leqslant |f(x) - f(y)| + |P_F(y) - P_F(x)| + |f(y) - P_F(y)|. \tag{7}$$

Since ϕ has been normalized so that $d(x, y) \geqslant |x - y|$, the first term in (7) is bounded above by $\omega_f(\delta)$. To bound the second term in (7), note that the function $P_F \circ \phi$ belongs to the subspace $\pi_n \circ \phi$ on which the derivative D has norm K_n. From the mean value theorem,

$$\begin{aligned} |P_F(y) - P_F(x)| &= |P_F(\phi(\phi^{-1}(x))) - P_F(\phi(\phi^{-1}(y)))| \\ &= K_n(\|P_F \circ \phi\|_{[a,b]})\, |\phi^{-1}(x) - \phi^{-1}(y)| \\ &< K_n \|P_F\|\, \delta. \end{aligned}$$

Then $\|f-P_F\| - \|f-P\| \leqslant \|f-P_F\| - \|f-P\|_F \leqslant \|f-P_F\| - \|f-P_F\|_F \leqslant \|f-P_F\| - |f(y)-P_F(y)|$, and from inequality (7) and the bounds on its first two terms,

$$\|f-P_F\| - \|f-P\| \leqslant w_f(\delta) + K_n\,\delta\,\|P_F\|. \tag{8}$$

Now to bound $\|P_F\|$: consider a polynomial Q in π_n which attains its norm on $[-1, 1]$ at a point x, and choose y in F with $d(x, y) < \delta$. Then $\|Q\| \leqslant |Q(x) - Q(y)| + |Q(y)|$, and

$$\|Q\| \leqslant |(Q \circ \phi)(\phi^{-1}(x)) - (Q \circ \phi)(\phi^{-1}(y))| + \|Q\|_F. \tag{9}$$

As above, the first term in (9) is bounded by $K_n\,\delta\,\|Q\|$. If δ is small enough to have $\delta K_n < 1$, it follows that

$$\|Q\| \leqslant \frac{1}{1 - K_n\delta}\,\|Q\|_F. \tag{10}$$

For $Q = P_F$, $\|Q\|_F \leqslant \|P_F - f\|_F + \|f\|_F \leqslant 2\,\|f\|_F \leqslant 2\,\|f\|$, and

$$\|P_F\| \leqslant 2\,\|f\|/(1 - K_n\delta). \tag{11}$$

Thus

$$\|f-P_F\| - \|f-P\| \leqslant w_f(\delta) + \frac{2K_n\delta}{1-K_n\delta}\,\|f\|,$$

and the inequality of the theorem follows from Eq. (6). Q.E.D.

THEOREM 2. *Let r, $0 < r < 1$, be given, and let F_n be a subset of $[-1, 1]$ for which $|F_n|\,K_n \leqslant 1 - r$. For any f in $C[-1, 1]$, let P_n be the polynomial in π_n which best approximates f on F_n. Then $\|f - P_n\| \leqslant (1 + 2/r)\,E_n(f)$, where, as usual, $E_n(f) = d(f, \pi_n)$.*

Proof. The proof is analogous to [5, p. 93]. Q.E.D.

The function ϕ, which has been fixed in Theorems 1 and 2, will now be varied and inequality (4) will be used to derive a criterion for choosing ϕ. Say that ϕ is *optimal* (with respect to inequality (4)) if, given $[a, b]$ and the number m of points in the finite subset F of $[-1, 1]$, ϕ minimizes the right-hand side of (4). Note that only the number m of points in F is given; F is otherwise unspecified.

If F contains m points $\{y_1, \dots, y_m\}$, then for a given ϕ, $|F|$ has its minimum value of $(b-a)/2m$ for the choice

$$y_j = \phi(a + (2j-1)(b-a)/2m), \qquad j = 1, 2, \dots, m. \tag{12}$$

This minimum value of $|F|$, for a given $[a, b]$, is independent of ϕ. Therefore, minimizing $K_n\delta$ by the optimal choice of ϕ, and thereby minimizing the right-hand side of inequality (4) as well as obtaining the best constant r in Theorem 2, is achieved by minimizing K_n. When the optimal ϕ has been found, Eq. (12) indicates the corresponding optimal choice for the m points in F.

A function B_n is a *bound for the derivative on* π_n if

$$|P'(x)| \leqslant B_n(x) \, \|P\|, \qquad \text{for all } P \text{ in } \pi_n, \ -1 < x < 1. \tag{13}$$

For a continuous bound B_n, define

$$C_n(x) = \int_{-1}^{x} B_n(t) \, dt. \tag{14}$$

The best bound B_n^*, which is continuous [23, p. 162], is given by

$$B_n^*(x) = \sup_P \{|P'(x)| : P \text{ in } \pi_n, \ \|P\| \leqslant 1\},$$

and the corresponding

$$C_n^*(x) = \int_{-1}^{x} B_n^*(t) \, dt.$$

For a discussion of B_n^* and related matters see [3, 23].

THEOREM 3. *Let B_n be a continuous bound on the derivative on π_n, and for a given interval $[a, b]$, define*

$$\phi_n(x) = C_n^{-1}\left(C_n(1)\left(\frac{x-a}{b-a}\right)\right). \tag{15}$$

Then

$$\inf_\phi \|D|_{\pi_n \circ \phi}\| \leqslant \|D|_{\pi_n \circ \phi_n}\| \leqslant C_n(1)/(b-a), \tag{16}$$

and for

$$\phi_n^*(x) = (C_n^*)^{-1}\left(C_n^*(1)\left(\frac{x-a}{b-a}\right)\right), \tag{17}$$

$$\inf_\phi \|D|_{\pi_n \circ \phi}\| = \|D|_{\pi_n \circ \phi_n^*}\| = C_n^*(1)/(b-a). \tag{18}$$

Proof. For P in π_n with $\|P\| \leqslant 1$, $|D(P \circ \phi)(x)| = |P'(\phi(x)) \, \phi'(x)| \leqslant |B_n(\phi(x)) \, \phi'(x)| = |DC_n(\phi(x))|$. The function $C_n \circ \phi$ maps $[a, b]$

monotonically onto $[0, C_n(1)]$; the choice of increasing ϕ which minimizes $\sup_x\{|DC_n(\phi(x))|: -1 \leqslant x \leqslant 1\}$ is given by (15) and (16) follows.

From the definition of the best bound B_n^*, given $\phi(x)$ in $(-1, 1)$ there is a P in π_n of norm one with

$$|D(P \circ \phi)(x)| = |B_n^*(\phi(x))\, \phi'(x)| = |DC_n^*(\phi(x))|.$$

Therefore,

$$\| D|_{\pi_n \circ \phi} \| = \{\sup_x |DC_n^*(\phi(x))|: -1 \leqslant x \leqslant 1\}.$$

As above, this norm is minimized by the function ϕ_n^* of Eq. (17). Q.E.D.

Equation (18) gives a formula for $K_n = \inf_\phi \| D|_{\pi_n \circ \phi} \|$. The best bound on the derivative on $n+1$-dimensional subspaces, $d_{n+1} = \inf_M \{\| D|_M \|: M$ a subspace in the domain of D, $\dim M = n+1\}$, is discussed in [24] where it is shown that $d_{n+1} = n$. From the asymptotic results given below (with $[a, b] = [-1, 1]$) $K_n / d_{n+1} \sim \pi/2$.

Given an interval $[a, b]$, K_n has a minimum value of $C_n^*(1)/(b-a)$. The interval $[a, b]$ is not relevant in minimizing the product $K_n \delta$; in fact, $K_n \delta$ has the value

$$\text{The minimum of } K_n \delta \text{ for } m \text{ points is } C_n^*(1)/2m \tag{19}$$

for the best choice (12) of m points, and this value does not depend on $[a, b]$. As (17) indicates, there is a family of optimal ϕ_n^* defined on different intervals and related by a linear change of variable.

Markov's inequality gives the bound $B_n(x) = n^2$ and, therefore, to within composition with a linear transformation, $\phi_n(x) = x$. For this ϕ_n the m points of (12) are equally spaced, and $K_n \delta \leqslant C_n(1)/2m = 2n^2/2m$.

Bernstein's inequality gives the bound $B_n(x) = n/\sqrt{1-x^2}$ and, to within composition with a linear transformation, $\phi_n(x) = \cos x$. For this ϕ_n the points (12) are the classical choice (1), and $K_n \delta \leqslant C_n(1)/2m = n\pi/2m$.

The best bound B_n^* gives the optimal ϕ_n^* of Eq. (17) and the smallest value $C_n^*(1)/2m$ for $K_n \delta$.

The formulas for B_2^* and B_3^* given in [3; 21, p. 112] allow the computation of $C_2^*(1) = 4.39...$ (compare with $2\pi = 6.28...$) and $C_3^*(1) = 7.02...$ (compare with $3\pi = 9.42...$). The function B_n^* is, in general, quite difficult to compute [3, 16, 23]. However it is possible to asymptotically estimate $C_n^*(1)$ and so compare the optimal ϕ_n^* with the classical $\cos x$ by comparing $C_n^*(1)$ and $n\pi$. To do this an elegant result of Bernstein's is needed:

$$B_n^*(x) \sim n/\sqrt{1-x^2}. \tag{20}$$

Bernstein's ideas in [1], after bypassing several difficulties in his proof, apply to give the asymptotic formula (20) with an error term which allows the calculation of an asymptotic formula for $C_n^*(1)$.

THEOREM 4. *Equation* (20) *holds; in fact for* $n \geqslant 4$,

$$0 \leqslant \frac{n}{\sqrt{1-x^2}} - B_n^*(x) \leqslant \frac{n^{1/2}}{\sqrt{1-x^2}} + \frac{7n^{1/4}}{(\sqrt{1-x^2})^2} + \frac{4}{n^{1/2}(\sqrt{1-x^2})^3} \tag{21}$$

and

$$\frac{C_n^*(1)}{n\pi} = 1 + O\left(\frac{1}{\sqrt{n}}\right). \tag{22}$$

Proof. Bernstein's basic idea is to consider

$$Q_n = \cos(n\theta - \delta), \qquad x = \cos\theta, \qquad 0 \leqslant \theta \leqslant \pi, \tag{23}$$

where δ is a temporarily unknown function of x. He was probably motivated to consider such a function by the extraordinary usefulness of the Chebyshev polynomials $T_n(x) = \cos n\theta$, $x = \cos\theta$. From the addition formula,

$$Q_n = \cos n\theta \cos\delta + \sin n\theta \sin\delta.$$

The trick is to choose δ so as to have a simple form for Q_n.

To motivate Bernstein's choice of δ, note that $\sin n\theta = \sqrt{1-x^2}\, U_{n-1}(x)$, U_{n-1} the Chebyshev polynomial of the second kind [21, p. 7]. Consider a right triangle with acute angle δ. The radical will be removed from the term $\sqrt{1-x^2}\, U_{n-1}(x) \sin\delta$ if the side opposite δ is choosen to be of length $k\sqrt{1-x^2}$. Then, letting y be the length of the side adjacent to δ, the denominator $\sqrt{y^2 + k^2(1-x^2)}$ of $\cos\delta$ will be simple if y is choosen to be a linear function of x which makes the radicand a perfect square. When this is done Bernstein's choice is obtained:

$$\delta = \arccos\left(\frac{ax-1}{a-x}\right), \qquad \sin\delta = \operatorname{sgn}(a)\sqrt{(a^2-1)(1-x^2)}/(a-x), \tag{24}$$

where a is a constant with $|a| > 1$ (and the sign of $\sin\delta$ is positive because $0 \leqslant \delta \leqslant \pi$). Then

$$Q_n(x) = [T_n(x)(ax-1) + U_{n-1}(x)(1-x^2)\operatorname{sgn}(a)\sqrt{a^2-1}\,]/(a-x) \tag{25}$$

is a polynomial of degree $n+1$ divided by $a-x$ and so has the form

$$Q_n(x) = P_n(x) + A/a - x, \tag{26}$$

P_n a polynomial of degree n and A a constant.

Use the formulas $T_n(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n]$, [21, p. 5] and $U_{n-1}(x) = (1/2\sqrt{x^2-1})[(x+\sqrt{x^2-1})^n - (x-\sqrt{x^2-1})^n]$ to compute

$$A = (a^2-1)(a-\operatorname{sgn}(a)\sqrt{(a^2-1)})^n. \tag{27}$$

As a function of $a \geqslant 1$, A attains its maximum when $a^2 = n^2/(n^2-4)$, and this maximum, which is asymptotic to $(4/e^2)(1/n^2)$, is bounded by $1/n^2$ for $n \geqslant 4$. Thus for $n \geqslant 4$ and $|a| > 1$,

$$|A| \leqslant 1/n^2. \tag{28}$$

The bound (28) shows that P_n, as given by (25) and (26), is, for large n, close to the function Q_n, and Q_n is a function which resembles a Chebyshev polynomial in the way it alternates between $+1$ and -1. Bernstein uses these ideas, and generalizations, to obtain several asymptotic results [2, p. 10–26].

Differentiate P_n:

$$P_n'(x) = \frac{n}{\sqrt{1-x^2}} \sin(n\theta - \delta)\left(1 - \frac{\sin\delta}{n\sqrt{1-x^2}}\right) - \frac{A}{(a-x)^2}. \tag{29}$$

Let x_0 in $(-1, 1)$ be given and set $\theta_0 = \arccos(x_0)$. For some integer k, $n\theta_0 - \pi/2 - k\pi$ is in $[0, \pi]$, and the range of the function $\delta_0(a) = \arccos((ax_0-1)/(a-x_0))$ on $|a| > 1$ is $(0, \theta_0) \cup (\theta_0, \pi)$. Consequently, given $\varepsilon > 0$ a value a' can be choosen with $|a'| > 1$ and $|\sin(n\theta_0 - \delta_0(a'))| \geqslant 1 - \varepsilon$. Suppose that $a' > 1$. If a' is too close to 1, $\|P_n\|$ will be too large; to get around this problem, take $a'' = \max(a', 1 + n^{-3/2})$, where the exponent 3/2 is choosen with the final asymptotic formula (22) in mind. If $a' < 1 + n^{-3/2}$, we need to estimate how close $\sin(n\theta_0 - \delta_0(a'))$ is to $\sin(n\theta_0 - \delta_0(1+n^{-3/2}))$. Since

$$\frac{d}{da}\cos(\delta_0(a)) = \frac{1-x_0^2}{(a-x_0)^2} \leqslant \frac{4}{1-x_0^2},$$

$$|\cos(\delta_0(a)) - \cos\delta_0(b)| \leqslant \frac{4\,|b-a|}{1-x_0^2}, \tag{30}$$

and it follows that

$$|\cos(\delta_0(a')) - \cos(\delta_0(1+n^{-3/2}))| \leqslant \frac{4}{n^{3/2}(1-x_0^2)}. \tag{31}$$

Set $a=1$ in (30) and multiply by $1-\cos(\delta_0(b))$ to get

$$\sin^2\delta_0(b) \leqslant \frac{8\,|b-1|}{(1-x_0^2)}.$$

Thus $\sin^2\delta_0(a')$ and $\sin^2\delta_0(1+n^{-3/2})$ are both bounded by $(8/n^{3/2})(1/(1-x_0^2))$. Then

$$\begin{aligned}
&|\sin(n\theta_0-\delta(a'))-\sin(n\theta_0-\delta_0(1+n^{-3/2}))| \\
&\quad =|\sin n\theta_0(\cos\delta_0(a')-\cos\delta_0(1+n^{-3/2}))+\cos n\theta_0(\sin\delta_0(1+n^{-3/2}) \\
&\qquad -\sin\delta_0(a'))| \\
&\quad \leqslant \frac{4}{n^{3/2}(1-x_0^2)}+\frac{2\cdot\sqrt{8}}{n^{3/4}\sqrt{1-x_0^2}}.
\end{aligned}$$

If ε is taken to be less than, say, $(6-2\sqrt{8})/n^{3/4}\sqrt{1-x_0^2}$, then

$$|\sin(n\theta_0-\delta_0(a''))| \geqslant 1-\frac{4}{n^{3/2}(1-x_0^2)}-\frac{6}{n^{3/4}\sqrt{1-x_0^2}} \tag{32}$$

holds for $a''=\max(a',1+n^{-3/2})$, when $a'>1$. When $a'<-1$, let $a''=\min(a',-1-n^{-3/2})$, and argue as above to see that (32) holds. Now set $a=a''$ in P_n.

From (23), (26), (28), and the fact that $|a''|\geqslant 1+n^{-3/2}$,

$$\|P_n\| \leqslant 1+\frac{|A|}{(|a''|-1)} \leqslant 1+\frac{1}{n^{1/2}}.$$

Using (29)

$$\begin{aligned}
|P_n'(x_0)| \geqslant{}& \frac{n}{\sqrt{1-x_0^2}}\,|\sin(n\theta_0-\delta_0(a''))|\left(1-\frac{1}{n\sqrt{1-x_0^2}}\right) \\
&-\frac{1}{n^2(1-|x_0|)^2}.
\end{aligned}$$

Using (32)

$$\begin{aligned}
|P_n'(x_0)| \geqslant{}& \frac{n}{\sqrt{1-x_0^2}}\left(1-\frac{1}{n\sqrt{1-x_0^2}}-\frac{4}{n^{3/2}(1-x_0^2)}-\frac{6}{n^{3/4}(\sqrt{1-x_0^2})}\right) \\
&-\frac{4}{n^2(1-x_0^2)} \\
={}& \frac{n}{\sqrt{1-x_0^2}}-\frac{1}{(1-x_0^2)}\left(1+6n^{1/4}+\frac{4}{n^2}\right)-\frac{4}{n^{1/2}(\sqrt{1-x_0^2})^3}.
\end{aligned}$$

For $n \geqslant 4$,

$$|P_n'(x_0)| \geqslant \frac{n}{\sqrt{1-x_0^2}} - \frac{7n^{1/4}}{1-x_0^2} - \frac{4}{n^{1/2}(\sqrt{1-x_0^2})^3}.$$

Hence $B_n^*(x_0) \geqslant |P_n'(x_0)|/\|P_n\| \geqslant (1 - 1/n^{1/2})\,|P_n'(x_0)|$ and (21) follows.

To estimate $\int_0^1 B_n^*(t)\,dt$, consider the point $\sqrt{1-1/n^2} = 1 - c_n$ where Markov's bound n^2 (on the derivative on π_n) and Bernstein's bound $n/\sqrt{1-x^2}$ agree, and break the interval of integration into $[0, 1-c_n]$ and $[1-c_n, 1]$ with $1 - c_n \sim 1 - 1/2n^2$. Then integrating Eq. (21) shows that

$$\int_0^{1-c_n} B_n^*(t)\,dt \sim n\pi/2 + O(\sqrt{n}),$$

while

$$\int_{1-c_n}^1 B_n^*(t)\,dt \leqslant \int_{1-c_n}^1 n^2\,dt \sim \tfrac{1}{2}.$$

Hence

$$\int_0^1 B_n^*(t)\,dt = n\pi/2 + O(\sqrt{n})$$

and (22) follows. Q.E.D.

References

1. S. Bernstein, Remarques sur l'inégalité de Wladimir Markoff, *Soobšč. Har'kov Mat. Obsc.* **14** (1913), 81–87.
2. S. Bernstein, Leçons son les Propertiés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle, *in* "L'Approximation," Chelsea, New York, 1970.
3. R. P. Boas, Jr., Inequalities for the derivatives of polynomials, *Math. Mag.* **42** (1969), 165–174.
4. B. A. Chalmers, On the rate of convergence of discretization in Chebyshev approximation, *SIAM J. Numer. Anal.* **15** (1978), 612–617.
5. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
6. P. C. Curtis, Convergence of approximating polynomials, *Proc. Amer. Math. Soc.* **13** (1962), 385–387.
7. C. B. Dunham, Approximation by alternating families on subsets, *Computing* **9** (1972), 261–265.
8. C. B. Dunham, Efficiency of Chebyshev approximation on finite subsets, *J. Assoc. Comput. Mach.* **21** (1974), 311–313.

9. C. B. DUNHAM AND J. WILLIAMS, Rate of convergence of discretization in Chebyshev approximation, *Math. Comp.* **37** (1981), 135–139.
10. H. ERLICH AND K. ZELLER, Schwankung von Polynomen, zwischen Gitterpunkten. *Math. Z.* **86** (1964), 41–44.
11. B. KRIPKE, Best approximation with respect to nearby norms, *Numer. Math.* **6** (1964), 103–105.
12. A. KROO, A comparison of uniform and discrete polynomial approximation, *Anal. Math.* **5** (1979), 35–50.
13. A. KROO, A note on discrete Chebyshev approximation, *Acta Math. Acad. Sci. Hungar.* **36** (1980), 129–135.
14. T. S. MOTZKIN AND J. L. WALSH, Least *p*th power polynomials on a real finite point set, *Trans. Amer. Math. Soc.* **83** (1956), 371–396.
15. I. P. NATHANSON, "Constructive Theory of Functions," Book 2, AEC translation #4503, National Technical Information Service, Springfield, V., 1961.
16. S. PASZKOWSKI, "The Theory of Uniform Approximation I. Nonasymptotic Theoretical Problems," Rozprawy Matematyczne XXVI, Warsawa, 1962.
17. J. PEETRE, Approximation of norms, *J. Approx. Theory* **3** (1970), 243–260.
18. J. RICE, 'The Approximation of Functions, I," Addison–Wesley, Reading, Mass., 1964.
19. J. RICE, "The Approximation of Functions, II," Addison–Wesley, Reading, Mass., 1969.
20. T. J. RIVLIN, "An Introduction to the Approximation of Functions," Blaisdell, Waltham, Mass., 1969.
21. T. J. RIVLIN, "The Chebyshev Polynomials," Wiley, New York, 1974.
22. T. J. RIVLIN AND E. W. CHENEY, A comparison of uniform approximations on an interval and a finite subset thereof, *J. SIAM Numer. Anal.* **3** (1966), 311–320.
23. E. V. VORONOVSKAJA, "The Functional Method and Its Applications," AMS Trans. Math. Monograph No. 28, Amer. Math. Soc., Providence, R. I., 1970.
24. R. WHITLEY, Markov and Bernstein's inequalities and compact and strictly singular operators, *J. Approx. Theory* **34** (1982), 277–285.